[Lecture 2 on 2021-05-19]

brainstorm: → "complete the square" (*)

Example 3.3:

How would you find a formula for the following indefinite integral?

$$\int \frac{dx}{x^2 - x + 1}$$

(sketch of the full solution)

→ make it look like a $\tan^{-1}(ax)$ antiderivative

→ the complete the square method: $(x-b)^2 = x^2 - 2b + b^2$, here $-2b = -1$ $\Rightarrow b = 1/2$

→ we get that: $x^2 - x + 1 = (x - 1/2)^2 + 3/4$

→ factor out 3/4 from the denominator

→ apply formula: $\int \frac{dx}{1+(ax)^2} = \frac{1}{a} \tan^{-1}(ax) + C$

Section 5.1-5.3: Area under the curve and the definite integral

Math 1552 lecture slides adapted from the course materials
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[Continuation of Lecture 2 on 2021-05-19]

Average Value

The ***average value*** of f on $[a,b]$ is the y-value that would generate a rectangle with the same area as f on $[a,b]$.

$$AV = \frac{Area}{\underbrace{[b-a]}_{\text{width of the interval } [a,b],\ a<b}}$$

Later:

$$AV = \frac{1}{b-a}\int_a^b f(t)dt$$

Example 3:

Using the same example, find the average value of the function on the interval [-1,2] using the midpoint estimate.

→ Recall: With $f(x) = \frac{1}{1+x^2}$ (and $N=6$), we obtained

Numerically That

$$\int_{-1}^{2} f(x)\,dx \approx \underbrace{M_f = \frac{430256}{226525} \approx 1.89938}_{\text{our "midpoint estimate" of the definite integral of } f \text{ on } [-1,2]} \;(!)$$

means "is approximated by"

→ recall that the avg. value (AV) is given by $AV = \dfrac{\text{Area}}{b-a}$

→ here, $AV \approx \dfrac{M_f}{3} = \dfrac{1}{3} \cdot \dfrac{430256}{226525}$

Next Learning Goals

- Be able to find the equation for a general Riemann Sum (as $\Delta x \to 0$)
- Take the limit of your answer to find the actual area beneath the curve
- Understand the definition of the definite integral
- Understand key properties of the definite integral

General Riemann Sum

Partition the interval $[a,b]$ into n equal pieces : $\to n \geq 1$

$a = x_0 < x_1 < x_2 < .... < \cancel{a_n}\,x_n = b$

Let x_i^* be an arbitrary point in the interval $[x_{i-1}, x_i]$. ex: x_1^* is a point in the subint. $[x_0, x_1]$

Then we can estimate the area under the curve

between $x = a$ and $x = b$ with the formula :

$$A \approx \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height of rect.}} \underbrace{\Delta x}_{\text{width of rect.}}$$

Note that : $L_f \leq A \leq U_f$

What is x_i^* in the formula?

$$A \approx \sum_{k=1}^{N} f(x_k^*)\Delta x$$

A. The left-hand endpoint of the subinterval.

B. The right-hand endpoint of the subinterval.

C. The midpoint of the subinterval.

D. Any value on the subinterval.

Trick Q

Same as letting $N \to \infty$

$\Rightarrow$ As $\Delta x \to 0$,

L_f, U_f, M_f are all going to converge to the same value, the exact area under the curve

The Definite Integral

We define the definite integral to be the limit of the Riemann Sum:

$$\boxed{\underbrace{\int_a^b f(x)dx}_{} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

(important formula to know)

The Definite Integral and Area

If the function is always non-negative on [*a*,*b*], we have found ***TOTAL AREA*** under the curve.

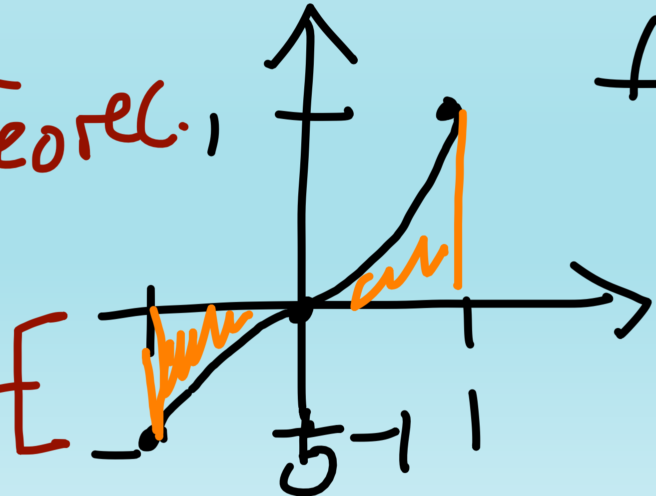
Compare to:

$$\int_a^b |f(x)|dx$$

If the function takes on negative values, then we have found the ***NET AREA*** under the curve.

$$\int_a^b f(x)dx$$

typo in the extra video rec.



$f(x)=x^3$

$\rightarrow \int_{-1}^{1} x^3 dx = 0$

Helpful Summation Formulas (memorize)

know these

$$\sum_{i=1}^{n} 1 = n \qquad (*)$$

$$\underbrace{1+1+\cdots+1}_{N \text{ times}} = N$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (*)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

know how to use these properties

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

(Linearity – use to simplify sums)

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

(Linearity – use to simplify sums)

Example 4:

Use the method of Riemann Sums to evaluate the following definite integral. Choose x_i^* to be the left-hand endpoint of each subinterval.

$$\int_{-1}^{2} (x+1)^2\, dx$$

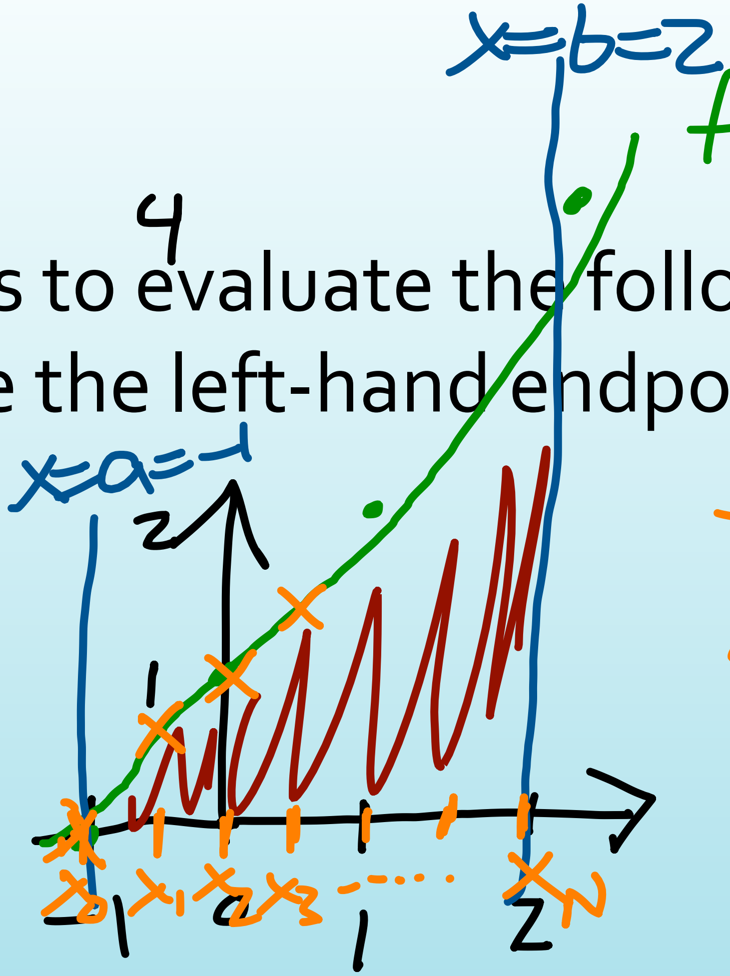
$f(x) = (x+1)^2$,

$[a,b] = [-1,2]$

$\to \Delta x = \dfrac{b-a}{N} = \dfrac{3}{N}$

$\to x_k^* = a + (k-1)\Delta x$

$= -1 + (k-1)\cdot \dfrac{3}{N}$



→ the RHS of a parabola (that is zero at x = −1)

$\rightarrow f(x_k^*) = (k-1)^2 \cdot \frac{9}{N^2} = \frac{9}{N^2}(k^2 - 2k + 1)$

<u>Procedure:</u>

$\rightarrow$ fix $N \geq 1$, and define

$$R_f(N) = \sum_{k=1}^{N} f(x_k^*)\Delta x$$

$$= \frac{27}{N^3}\left[\sum_{k=1}^{N} k^2 - 2\cdot\sum_{k=1}^{N} k + \sum_{k=1}^{N} 1\right]$$

$$= \frac{27}{N^3}\left[\frac{N(N+1)(2N+1)}{6} - 2\cdot\frac{N(N+1)}{2} + N\right] \quad (*)$$

$\rightarrow \int_{-1}^{2} (x+1)^2 dx = \lim_{N\to\infty} R_f(N)$

$\boxed{\text{NOTE}}$

$\rightarrow \lim_{N\to+\infty} \frac{c_1 N, N^2}{N^3} = 0, \quad \lim_{N\to\infty} \frac{aN^3}{N^3} = a$

$\rightarrow$ So by expanding out the terms from $(*)$, we get that:

$$\int_{-1}^{2} (x+1)^2 dx = \lim_{N\to\infty} R_f(N) = \frac{27 \cdot 2}{6} = 9$$

✓ :)

Example 5:

In a memory experiment, the rate of memorization is measured by the function:

approx shape of f:

$$f(t) = -c\,t^2 + d\,t,\ c>0,\ d>0,$$

where t is the time in minutes, and $f(t)$ is the number of words per minute.

(a) How many words are memorized in the first 2 minutes (from $t=0$ to $t=2$)? USE RIEMANN SUMS.

(b) What is the *average* number of words memorized each minute?

(AV of f ON [0,2])

(a) $f(x) = -cx^2 + dx, \quad [a,b] \equiv [0,2]$

$$\rightarrow \Delta x = \frac{b-a}{N} = \frac{2}{N}$$

$$x_k^* = 0 + k\Delta x = \frac{2k}{N}$$

$$\rightarrow R_f(N) = \sum_{k=1}^{N} f(x_k^*)\,\Delta x$$

$$= \frac{2}{N}\left[-c \cdot \sum_{k=1}^{N} \overbrace{\frac{4k^2}{N^2}}^{(x_k^*)^2} + d \cdot \sum_{k=1}^{N} \overbrace{\frac{2k}{N}}^{x_k^*}\right]$$

Cancel out

$$= \frac{2}{\boxed{N}}\left[-\frac{4c}{N^2}\cdot\frac{\cancel{N}(N+1)(2N+1)}{6} + \frac{2d}{\cancel{N}}\cdot\frac{\cancel{N}(N+1)}{2}\right]$$

$$= -\frac{8c}{6N^2}(N+1)(2N+1) + \frac{4d}{N}\frac{(N+1)}{2}$$

$$= -\frac{4c}{3N^2}(N+1)(2N+1) + \frac{2d}{N}(N+1) \quad (*)$$

$$\to \int_0^2 f(t)\,dt = \lim_{N\to\infty} R_f(N)$$

$$= -\frac{8c}{3} + 2d, \text{ by } (*)$$

(b) avg. # words memo. each minute

$$= AV = \boxed{\frac{1}{b-a}} \boxed{\int_a^b f(t)\,dt}$$

$$= \frac{1}{2} \cdot -\frac{8c}{3} + 2d$$

$$= \frac{-4c}{3} + d \quad \checkmark$$

Properties of the Definite Integral

Let $f(x)$ be continuous on $[a,b]$.

$$(1)\int_a^b c\,dx = c(b-a) \Longrightarrow$$

FTC interpret.:

$$c\int_a^b dx = c\cdot x\Big|_{x=a}^{x=b} = c(b-a)$$

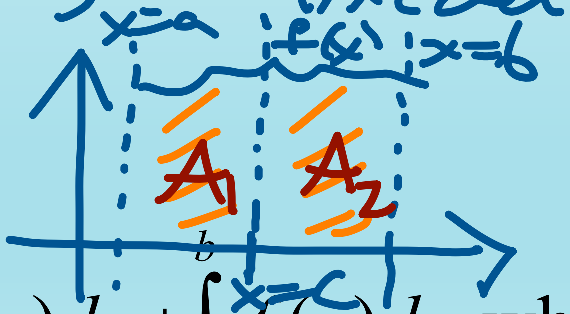
$$(2)\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(3)\int_a^a f(x)dx = 0$$

$$(*)\ (4)\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ where } \underline{c\in[a,b]}$$

$c\in(a,b)$

geometrically: $A = \int_a^b f(t)dt$



$$\Rightarrow A = A_1 + A_2, \text{ where}$$

$$A_1 = \int_a^c f(x)dx,$$

$$A_2 = \int_c^b f(x)dx$$

Properties of the Definite Integral (cont.)

linearity

(important to know how to apply)

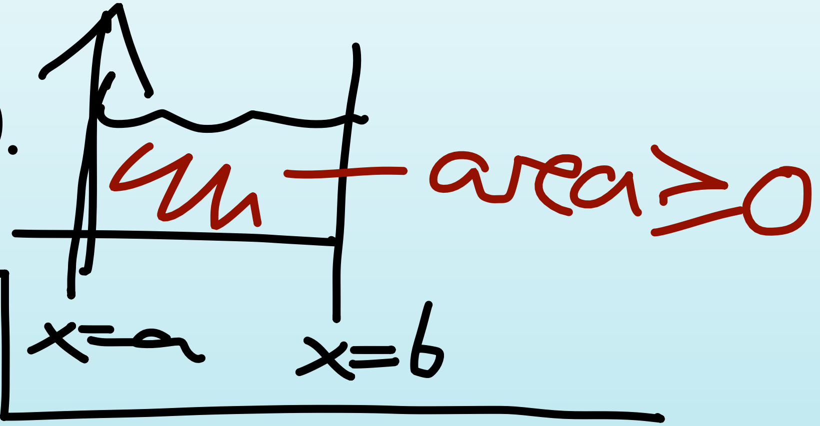
$$(5)\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$(6)\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Some More Integral Properties

(1) If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$.

(on $[a,b]$)



(2) If $f(x) \geq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

(3) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

More Integral Properties (cont.)

(4) If f is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0.$$

→ odd func:
$f(x) = -f(-x)$,
for all $x > 0$

ex: $f(x) = x^3$, $\sin(x)$

(5) If f is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

→ even func:
$g(x) = g(-x)$,
for all $x > 0$

ex: $g(x) = x^2$, $\cos(x)$

Example 6:

Given that $\int_1^3 2f(x)dx = 4$ and $\int_1^0 f(x)dx = -1$,

find $\int_0^3 f(x)dx. = I$

$$I = \int_1^3 f(x)dx + \int_0^1 f(x)dx$$

$$= \frac{1}{2}\int_1^3 2f(x)dx - \int_1^0 f(x)dx$$

$$= \frac{1}{2}\cdot 4 - (-1) = 3$$

A. -3

B. 1

C. $\boxed{3}$

D. 5

Challenge Problem:

Hints:

(1) What are some ways we have seen to simplify this integral?

(2) Recall your special triangle values of trig functions.

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(x^5 + 10x^3 + x^2 + 3x + 1)}{(x^2+1)^2} dx = ?$$

(H1) Notice that $g(x) = \frac{x^5 + 10x^3 + 3x}{(x^2+1)^2}$ is ODD

why? for $a > 0$, $g(a) = \dfrac{a^5+10a^3+3a}{(a^2+1)^2}$

$$= -g(-a) = -\frac{\left[(-a)^5+10(-a)^3-3a\right]}{((-a)^2+1)^2}$$

$$\Rightarrow \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(x^5+10x^3+3x)}{(x^2+1)^2}\,dx = 0$$

$$\rightarrow \text{So, } I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(x^2+1)\,dx}{(x^2+1)^2} = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

→ recall: $\displaystyle\int \frac{dx}{x^2+1} = \tan^{-1}(x) + C$

→ preview of FTC (on Friday):

[(HZ): $\tan(\pm\pi/3) = \pm\sqrt{3}$] (*)

$I = \tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3})$

$= \dfrac{\pi}{3} - \left(\dfrac{-\pi}{3}\right)$, by (*)

$= \dfrac{2\pi}{3}$

See you all on Friday! :)

(reminder to come to office hours:
MW @ 3-4PM and
Fri @ 1-2PM on BlueJeans)